

MATH 2050 C Lecture 9 (Feb 14)

Quiz 2 on Feb 16, covers up to Lecture 8.

Last week ϵ - K defⁿ for limits & some examples

Questions

- ① $\lim(x_n)$ exists?
- ② How to compute $\lim(x_n)$?

§ Limit Theorems (Textbook § 3.2)

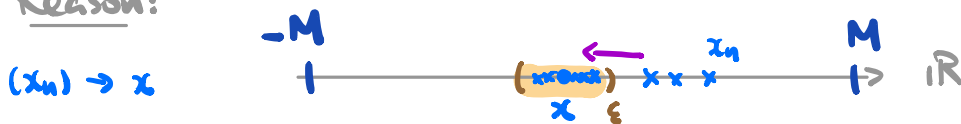
Defⁿ: (x_n) is bounded if $\exists M > 0$ st. independent of n !

$$|x_n| \leq M \quad \forall n \in \mathbb{N}$$

Remark: This is equivalent to $\{x_n : n \in \mathbb{N}\}$ is bounded (as a set).

Thm: (x_n) convergent \Rightarrow (x_n) bounded

Reason:



Proof: Since (x_n) is convergent, by defⁿ, $\exists x \in \mathbb{R}$ st.

$$\lim(x_n) = x.$$

$$\left[\begin{array}{l} \text{ie } \forall \epsilon > 0, \exists K \in \mathbb{N} \text{ st} \\ |x_n - x| < \epsilon \quad \forall n \geq K \end{array} \right]$$

Take $\epsilon = 1$, by defⁿ of limit, $\exists K \in \mathbb{N}$ st.

$$|x_n - x| < \epsilon = 1 \quad \forall n \geq K$$

By Triangle ineq., $\forall n \geq K$.

$$|x_n| = |(x_n - x) + x| \leq |x_n - x| + |x| < 1 + |x|$$

Choose $M := \max\{|x_1|, |x_2|, \dots, |x_{k-1}|, 1 + |x|\} > 0$

Then, $|x_n| \leq M \quad \forall n \in \mathbb{N}$

_____ \square

Remark: The converse of this thm can be used to prove that a sequence is divergent.

i.e. (x_n) unbdd $\Rightarrow (x_n)$ divergent.

Example: $(x_n) := (n)$ unbdd, hence divergent.

Caution: (x_n) bdd $\not\Rightarrow (x_n)$ convergent

(we'll return to this later.)

Recall: \mathbb{R} is a complete ordered field.

↑ ↑ ↑
(kind of) compatible

Limit Theorems: Suppose $\lim(x_n) = x$, $\lim(y_n) = y$. Then.

(i) $\lim(x_n \pm y_n) = x \pm y$

(ii) $\lim(x_n y_n) = x y$

(iii) $\lim\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$. provided $y_n \neq 0 \quad \forall n \in \mathbb{N}$ and $y \neq 0$

(i.e. the limits exist & are equal to the "expected" value.)

Proof: (i) Let $\varepsilon > 0$ be fixed but arbitrary.

Since $\lim(x_n) = x$ and $\lim(y_n) = y$,

$\exists k_1, k_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2} \quad \forall n \geq k_1$$

$$\text{and } |y_n - y| < \frac{\varepsilon}{2} \quad \forall n \geq k_2$$

Choose $K := \max\{k_1, k_2\} \in \mathbb{N}$, then $\forall n \geq K$, we have

$$\begin{aligned} |(x_n \pm y_n) - (x \pm y)| &\stackrel{\text{triangle inequality}}{\leq} |x_n - x| + |y_n - y| \\ &\stackrel{\text{since } n \geq k_1, k_2}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

... 2

Goal: Find K s.t.
 $|(x_n + y_n) - (x + y)| < \varepsilon$
in
 $|(x_n - x)| + |(y_n - y)|$
small small
 $\because x_n \rightarrow x \quad \because y_n \rightarrow y$

(ii) Let $\varepsilon > 0$ be fixed but arbitrary.

Since (y_n) is convergent, by previous thm.

(y_n) is bdd, i.e. $\exists M > 0$ s.t.

$$|y_n| \leq M \quad \forall n \in \mathbb{N}$$

Take $M' := \max\{M, 1 + |x|\} > 0$

By defⁿ of limit (taking $\varepsilon = \varepsilon / 2M' > 0$)

then $\exists k_1, k_2 \in \mathbb{N}$ s.t.

$$|x_n - x| < \frac{\varepsilon}{2M'} \quad \forall n \geq k_1$$

$$\text{and } |y_n - y| < \frac{\varepsilon}{2M'} \quad \forall n \geq k_2$$

Choose $K := \max\{k_1, k_2\} \in \mathbb{N}$, then $\forall n \geq K$, we have

... 1

Goal: Find K s.t. $\forall n \geq K$
 $|x_n y_n - xy| < \varepsilon$

Estimate:
 $|x_n y_n - xy|$
 $= |x_n y_n - x y_n + x y_n - xy|$
 $\leq |y_n(x_n - x)| + |x(y_n - y)|$
 $= |y_n| |x_n - x| + |x| |y_n - y|$
bold by Thm small ? fixed number small
 $\leq M |x_n - x| + M |y_n - y| \stackrel{\varepsilon/2M'}{\leq} \varepsilon$

$$\begin{aligned}
|x_n y_n - xy| &= | \underbrace{x_n y_n - x y_n}_{\text{red}} + \underbrace{x y_n - xy}_{\text{red}} | \\
&= | y_n (x_n - x) + x (y_n - y) | \\
&\leq |y_n| |x_n - x| + |x| |y_n - y| \\
&\leq M |x_n - x| + |x| |y_n - y| \\
&\leq M' |x_n - x| + M' |y_n - y| \\
&< M' \cdot \frac{\epsilon}{2M'} + M' \cdot \frac{\epsilon}{2M'} = \epsilon
\end{aligned}$$

(iii) Since $\left(\frac{x_n}{y_n}\right) = \left(x_n \cdot \frac{1}{y_n}\right)$, using (ii), it suffices to show

(#): $\lim \left(\frac{1}{y_n}\right) = \frac{1}{y}$ provided $y_n \neq 0 \forall n \in \mathbb{N}$ and $y \neq 0$.

Let $\epsilon > 0$ be fixed but arbitrary.

We first establish a lemma.

Lemma: $\exists \tilde{K} \in \mathbb{N}$ st

$$|y_n| \geq \frac{|y|}{2} \quad \forall n \geq \tilde{K}$$

Pf of Lemma: Since $\lim (y_n) = y$,

by take $\epsilon := \frac{|y|}{2} > 0$ since $y \neq 0$.

$$\exists \tilde{K} \in \mathbb{N} \text{ st } |y_n - y| < \epsilon = \frac{|y|}{2}$$

$$\forall n \geq \tilde{K}$$

By reverse triangle ineq, $\forall n \geq \tilde{K}$.

$$\begin{aligned}
|y_n| &= |y + (y_n - y)| \geq ||y| - |y_n - y|| \\
&\geq |y| - \frac{|y|}{2} = \frac{|y|}{2}
\end{aligned}$$

Goal: Find K st $\forall n \geq K$

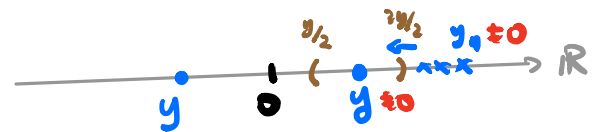
$$\left| \frac{1}{y_n} - \frac{1}{y} \right| < \epsilon$$

Estimate: (want: $|y_n - y|$)

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y - y_n}{y_n y} \right|$$

$$= \frac{|y_n - y|}{|y_n| |y|}$$

$$= \underbrace{\left(\frac{1}{|y_n|}\right)}_{\text{fixed no.}} \cdot \underbrace{\frac{1}{|y|}}_{\text{fixed no.}} \cdot \underbrace{|y_n - y|}_{\text{small}}$$



Lemma: $|y_n| \geq \frac{|y|}{2} > 0 \quad \forall n \geq \tilde{K}$

Since $\lim(y_n) = y$, taking $\frac{\varepsilon}{2|y|^2} > 0$.

$$\exists K' \in \mathbb{N} \text{ st } |y_n - y| < \frac{\varepsilon}{2|y|^2} \quad \forall n \geq K' \quad \leftarrow (*)$$

Choose $K := \max\{\tilde{K}, K'\} \in \mathbb{N}$. then $\forall n \geq K$, we have

$$\left| \frac{1}{y_n} - \frac{1}{y} \right| = \left| \frac{y_n - y}{y_n y} \right| = \frac{1}{|y_n|} \cdot \frac{1}{|y|} |y_n - y|$$

$$\leq \boxed{\frac{1}{|y|/2}} \cdot \frac{1}{|y|} \cdot \underbrace{\frac{\varepsilon}{2|y|^2}}_{(*) \text{ choice of } K'} = \varepsilon$$

□

Non-example: The assumptions in (iii) are necessary.

Consider $(y_n) := (\frac{1}{n}) \rightarrow y = 0$, then

$(\frac{1}{y_n}) = (n)$ is divergent.

Remark: Converse of the Thm is not true.

Eg. $(x_n) = (\frac{1}{n})$ convergent to 0, $(y_n) = (n)$ divergent then $(x_n y_n) = (1)$ convergent

Thm: Let $(x_n), (y_n)$ be two sequences of real numbers st.

$$x_n \leq y_n \quad \forall n \in \mathbb{N} \quad \text{---} \quad (**)$$

THEN, $\lim(x_n) \leq \lim(y_n)$ provided that their limits exist.

Remarks: ⁽ⁱ⁾ For $(**)$, it is also sufficient to have

$$x_n \leq y_n \quad \forall n \geq L \quad \text{for some fixed } L.$$

(ii) Even if we assume $x_n < y_n \quad \forall n \in \mathbb{N}$ in $(**)$,

we still get $\lim(x_n) \leq \lim(y_n)$ only.

E.g.) $0 < \frac{1}{n} \quad \forall n \in \mathbb{N}$ But $\lim(0) = 0 = \lim(\frac{1}{n})$.

Proof of Thm: By Limit Thm (i), it suffices to show

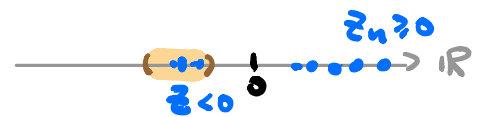
(*): (z_n) st $z_n \geq 0 \quad \forall n \in \mathbb{N} \Rightarrow \lim(z_n) =: z \geq 0$.
Conv. Seq.

Suppose NOT., then $z := \lim(z_n) < 0$.

Take $\varepsilon = \frac{|z|}{2} > 0$, then $\exists K \in \mathbb{N}$ st

$$|z_n - z| < \varepsilon = \frac{|z|}{2} \quad \forall n \geq K.$$

$$\Rightarrow z_n < z + \frac{|z|}{2} = -\frac{|z|}{2} < 0 \quad \forall n \geq K.$$



Contradicting $z_n \geq 0 \quad \forall n \in \mathbb{N}$!